

Guaranteed Tracking Controller for Wheeled Mobile Robot Based on Flatness and Interval Observer

Amine Abadi, Adnen El Amraoui, Hassen Mekki, Nacim Ramdani

Abstract—This paper proposes a guaranteed tracking controller for a Wheeled Mobile Robot (WMR) based on the differential flatness theory and the interval observer. Using the flatness property, it is possible to transform the non linear WMR model into a canonical Brunovsky form, for which it is easier to create a state feedback controller. Since, in most real applications, the WMR is subjected to uncertainties such as slip, disturbance and noise, control algorithms must be modified to take into account those uncertainties. Therefore, based on the information of the upper and lower limits of the initial condition and all the uncertainties, an interval observer that generates an envelope enclosing every feasible state trajectory is developed. After that, based on the center of the obtained interval observer, a new control law is proposed to guarantee the tracking performance of the WMR despite the existence of un-measurable states and bounded uncertainties. The closed-loop stability of the system is proven analytically using the Lyapunov theorem. A lot of numerical simulation is realized in order to demonstrate the efficiency of the suggested guaranteed tracking control scheme.

I. INTRODUCTION

In the last decade, special attention has been paid to mobile robots in view of their particular structure, automatic programming and practical challenges. The application domain of these systems has become very varied, such as national defense, hospital tasks, logistics industry and other areas. In order to accurately realize these different practical tasks, a highly performant guidance scheme needs to be proposed for the mobile robot.

Recently, the differential flatness property introduced by Fliess [1] has proven to be a good tool to ameliorate the trajectory planning and to create tracking controllers for linear and nonlinear systems. Thus, with flatness, all the state and control inputs of the system can be written as a function of the flat outputs and their derivatives. This property allows us to eliminate the utilization of the complex integration process. In addition, the flatness has specific advantages when used in nonlinear control systems. Indeed, by permitting an accurate linearization of the system's dynamical model, it can be possible to avoid utilizing the linear models with limited validity in the controller design. This characteristic makes

flatness a good tool for solving various problems in many areas [2-4].

In the last decade, the flatness property has been extensively used for the planning and tracking of mobile robots. In [5], Abadi put forward an optimal trajectory generation algorithm for Wheeled Mobile Robot (WMR) based on the transcription method, flatness and the B-spline curve. In [6], Luviano combined the flatness feedforward control and the generalized proportional integral to ensure good trajectory tracking for the WMR. In [7], Nasr proposed higher coverage trajectory generation strategies of a WMR utilizing the flatness property.

In the majority of studies on the tracking control of the WMR, nonholonomic assumption has been utilized. This leads to neglecting the existence of slip between the ground and the wheels. However, in real applications, slip is caused by many reasons such as high speed and uneven terrain. Generally, the slippage is not the only problem that can affect the tracking task of the WMR because most of the controllers and observers applied to this latter are based on the assumption that all the states of the system are available and that the disturbances and the measurement noise are negligible. Consequently, a new guidance law is necessary to be developed in order to guarantee the tracking result of the WMR subjected to unknown but bounded uncertainties (slippage, disturbance, measurement noise).

In the recent years, interval observers have become a robust approach to dealing with the state estimation problems for a system affected by disturbances and/or uncertainties, which are supposed to be unknown but bounded. This theory was originally introduced in [8-9] and has been successfully applied in several applications [10-13]. Starting from the knowledge of the upper and lower limits of the initial conditions and the uncertainties, interval observers can be developed to produce upper and lower bounds of state variables of dynamical systems at every time instant. Accordingly, the bounds offer intervals where the estimated variables are sure to stay for transient periods during which the classical observers cannot ensure any guarantee. One of the important conditions to create interval observers treated the cooperativity of the estimation error dynamics, which was relaxed in [14]. It was demonstrated that according to a constructive procedure, a Hurwitz matrix could be transformed to a Metzler and Hurwitz one (cooperative). The interval observer design for linear and nonlinear systems with model uncertainty was discussed in [15-16]. However, the existing results showed that the observation error would converge to an interval whose size depended on the value of

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 823887.

A. Abadi, is with University of Sousse, NOCCS Laboratory, Tunisia. and is with Univ. Orléans, INSA CVL, PRISME EA 4229, F45072 Orléans, France. amine.abadi@etu.univ-orleans.fr

H. Mekki is with University of Sousse, NOCCS Laboratory, Tunisia. mekki.hassen@gmail.com

A. El Amraoui and N. Ramdani are with Univ. Orléans, INSA-CVL, PRISME, EA 4229, F45072, Orléans, France. adnen4@gmail.com, nacim.ramdani@univ-orleans.fr

uncertainty. In addition, any state belonging to this interval could be considered as a robust state estimation for the system. Hence, compared to a punctual observer, the estimation interval guarantees more robustness when dealing with unknown bounded uncertainties. This property encourages us to exploit the advantage of the interval observer in practical applications such as tracking the trajectory of the WMR.

In this paper, our contribution consists in designing of guaranteed trajectory tracking control for the WMR despite the existence of unknown but bounded uncertainties, such as slippage, external environmental perturbation and measurement noise. Based on the differential flatness properties, it is possible to change the WMR equation model into a linear canonical (Brunovsky) form. For the obtained linearized system, it is simpler to develop a stabilizing feedback controller. To improve the tracking robustness of the WMR, an interval observer is developed to create an envelope containing all possible state estimations based on the upper and lower values of initial conditions and all uncertainties. Subsequently, the center of the interval observer is considered as a robust state estimation of the WMR. Finally, based on this robust estimation, flatness control, combined with an estimated feedback law, is developed in order to guarantee that the WMR tracks the reference trajectory in a precise interval. To the best of our knowledge, it is the first time in the literature that the advantages of interval estimation techniques are exploited in the design of tracking control for an uncertain WMR system.

This article is organized as follows. In section II, the WMR model is presented. In section III, the tracking controller for the WMR without considering any uncertainties is defined. In section IV, the uncertain model of the WMR is defined as well. In section V, the interval observer design is described. Section VI is devoted to the creation of guaranteed tracking control. Section VII deals with the simulation results, and section VIII concludes the paper.

II. WHEELED-ROBOT MOBILE MODEL

The system utilized in this paper consists of a two-wheeled differential driven mobile robot (Figure.1). It is equipped with two independently driven wheels (right and left) and a front wheel to ensure the equilibrium of the robot movement. The generalized configuration of the driven robot mobile is given by $q = [x; y; \theta]$, where x and y are the center position coordinates of the mobile robot in the fixed frame $(O; X; Y)$ and θ represents the robot orientation angle with respect to the X axis. The kinematic model of the WMR without slip is defined as follows:

$$\begin{aligned} \dot{x} &= \cos(\theta) v \\ \dot{y} &= \sin(\theta) v \\ \dot{\theta} &= w \end{aligned} \quad (1)$$

where v and w are the translational and rotational velocity of the robot, respectively. This latter can be written as a function of right and left angular speeds of the wheels (w_r

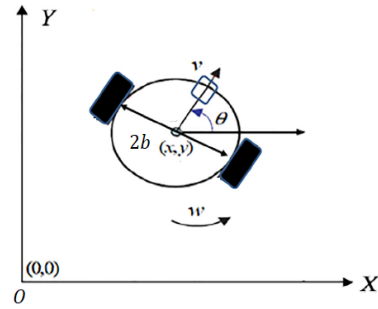


Fig. 1: Two-wheeled mobile robot

and w_l) as follows:

$$v = \left(\frac{w_r + w_l}{2} \right) r \quad (2)$$

$$w = \left(\frac{w_r - w_l}{2b} \right) r \quad (3)$$

where r is the radius of the wheel, and $2b$ is the distance between the wheels. According to the non slip condition, the non-holonomic constraint is defined as follows:

$$x \sin \theta - y \cos \theta = 0 \quad (4)$$

Equation (4) signifies that accurate tracking will be realizable only if the desired trajectories are feasible for the physical platform.

III. FLATNESS-BASED TRACKING CONTROL

The differential flatness concept represents a fundamental property that characterizes some nonlinear systems. It can be shown that the WMR is a differentially flat system, whose flat outputs are given by $z = [z_1; z_2] = [x; y]$. Therefore, all the states and the control of the WMR system can be written in terms of flat output and their derivatives as follows:

$$\theta = \arctan\left(\frac{-z_2}{-z_1}\right) \quad (5)$$

$$v = \frac{\dot{z}_1}{-z_1} + \frac{\dot{z}_2}{-z_2} \quad (6)$$

$$w = \frac{-\dot{z}_1 \cdot z_2}{-z_1} + \frac{\dot{z}_2 \cdot z_1}{-z_2} \quad (7)$$

The relationship between the control input vector, w and v , and the flat output's highest derivatives is not invertible. This problem obviously exposes an obstacle to realize static feedback linearization. To overcome this fact, the control input v is considered as an additional state for the kinematics model (1), consequently, the new extended system is defined as follows:

$$\begin{aligned} \dot{x} &= \cos(\theta) v \\ \dot{y} &= \sin(\theta) v \\ \dot{v} &= u_1 \\ \dot{w} &= u_2 \end{aligned} \quad (8)$$

where the state system of the mobile robot is $X = [x; y; v; \theta]^T$ and the new control input is defined by $u_1 = v$ and $u_2 = w$. The invertible relation between the inputs u_1

and u_2 and the higher derivatives of the flat outputs $y_{11} = x$ and $y_{21} = y$ is defined as follows:

$$\begin{aligned} \dot{u}_{11} &= B u_1 \\ \dot{u}_{21} &= B u_2 \end{aligned} \quad (9)$$

$$\text{with } B = \begin{pmatrix} \cos(\theta) & v \sin(\theta) \\ \sin(\theta) & v \cos(\theta) \end{pmatrix}$$

The matrix B is not singular if $v \neq 0$. Under this assumption, the control can be defined as follows:

$$\begin{aligned} u_1 &= B^{-1} \dot{u}_{11} \\ u_2 &= B^{-1} \dot{u}_{21} \end{aligned} \quad (10)$$

Thus, the model of WMR can be written in the two-linear canonical (Brunovsky) form as follows:

$$\begin{aligned} \dot{y}_{11} &= y_{12} & \dot{y}_{21} &= y_{22} \\ BF_1 \dot{y}_{12} &= v_x & BF_2 \dot{y}_{22} &= v_y \\ Y_1 &= y_{11} = x & Y_2 &= y_{21} = y \end{aligned} \quad (11)$$

where v_x and v_y are appropriate feedback controllers that permit the flat outputs y_{11} and y_{21} to track the desirable reference trajectories x_d and y_d , respectively. The feedback controllers are defined as follows:

$$v_x = \dot{u}_{x1} - K_{x2} e_{rx1} - K_{x1} e_{rx1} \quad (12)$$

$$v_y = \dot{u}_{y1} - K_{y2} e_{ry1} - K_{y1} e_{ry1} \quad (13)$$

where $e_{rx1} = y_{11} - x_d$, $e_{ry1} = y_{21} - y_d$, K_{x1} , K_{x2} , K_{y1} and K_{y2} are a controller gain that can be chosen so that the characteristic polynomial associated to each flat output tracking error is Hurwitz. The characteristic polynomials of the Burnovsky systems (11) are defined as follows:

$$s^2 + K_{x2}s + K_{x1} = s^2 + 2 \zeta_x \omega_{xc}s + \omega_{xc}^2 \quad (14)$$

$$s^2 + K_{y2}s + K_{y1} = s^2 + 2 \zeta_y \omega_{yc}s + \omega_{yc}^2 \quad (15)$$

where the parameters ζ_x and ζ_y are the damping coefficient, and ω_{xc} and ω_{yc} are the bandwidths of the controller. Based on equations (14) and (15), the controllers gain can be calculated as follows:

$$K_{x1} = \omega_{xc}^2; K_{x2} = 2 \zeta_x \omega_{xc}; K_{y1} = \omega_{yc}^2; K_{y2} = 2 \zeta_y \omega_{yc} \quad (16)$$

The Flatness-Based Tracking Control (FBTC) applied to the WMR can be obtained when replacing \dot{u}_{11} and \dot{u}_{12} by the feedback controllers v_x and v_y in the control input (10) as follows:

$$\begin{aligned} u_{FBTC1} &= B^{-1} \begin{pmatrix} \dot{u}_{x1} - K_{x2} e_{rx1} - K_{x1} e_{rx1} \\ \dot{u}_{y1} - K_{y2} e_{ry1} - K_{y1} e_{ry1} \end{pmatrix} \end{aligned} \quad (17)$$

The dynamics of the closed-loop tracking error of x and y are defined as follows:

$$\dot{e}_{rx} = H_{rx} e_{rx} \quad (18)$$

$$\dot{e}_{ry} = H_{ry} e_{ry} \quad (19)$$

with:

$$e_{rx} = [e_{rx1}; e_{rx1}]^T, e_{ry} = [e_{ry1}; e_{ry1}]^T;$$

$$H_{rx} = \begin{pmatrix} 0 & 1 \\ K_{x1} & K_{x2} \end{pmatrix}, H_{ry} = \begin{pmatrix} 0 & 1 \\ K_{y1} & K_{y2} \end{pmatrix}$$

The closed-loop stability of the error tracking systems can be ensured by appropriately choosing the controller poles.

IV. UNCERTAIN KINEMATIC MODEL

We consider that the WMR is subjected to three sources of uncertainties, namely as slippage, external environmental disturbance, and measurement noise. In this case, the Kinematic model of the WMR will be considered as follows:

$$\begin{aligned} \dot{x} &= \cos(\theta)v + v_t \cos(\theta) + v_s \sin(\theta) + d_x \\ \dot{y} &= \sin(\theta)v + v_t \sin(\theta) - v_s \cos(\theta) + d_y \\ \dot{\theta} &= w + w_s + d \end{aligned} \quad (20)$$

where d_x , d_y and d represent the external disturbances, v_t and v_s represent the slip velocities in the forward direction and normal to the forward direction, respectively, and w_s represents the angular slip velocity component. Based on [17], we assume that the slip component are defined as follows:

$$v_t(t) = v_s(t) = w_s(t) = \gamma_1 v(t) \quad (21)$$

where γ_1 is a positive constant.

We assume that the component velocity and the external disturbance and their derivatives are bounded as follows:

$$|v_t| \leq v_{tj}, |v_s| \leq v_{sj}, |w_s| \leq w_{sj}, |d_x| \leq d_{xj}, |d_y| \leq d_{yj}, |d| \leq d_j \quad (22)$$

$$|\dot{v}_t| \leq v_{tj}, |\dot{v}_s| \leq v_{sj}, |\dot{w}_s| \leq w_{sj}, |\dot{d}_x| \leq d_{xj}, |\dot{d}_y| \leq d_{yj}, |\dot{d}| \leq d_j \quad (23)$$

$$|d_x| \leq d_{xj}, |d_y| \leq d_{yj}, |d| \leq d_j \quad (24)$$

$$|d_x| \leq d_{xj}, |d_y| \leq d_{yj}, |d| \leq d_j \quad (25)$$

where v_{tj} , v_{sj} and w_{sj} $i = 1; \dots; 6$ are known values.

When choosing the control input defined by equation (10) and considering that WMR is subjected to slippage, disturbance and noise, the two integral chains of the Burnovsky form (11) are modified as follows:

$$\begin{aligned} \dot{y}_{11} &= y_{12} & \dot{y}_{21} &= y_{22} \\ MBF_1 \dot{y}_{12} &= v_x + \zeta_x & MBF_2 \dot{y}_{22} &= v_y + \zeta_y \\ Y_1 &= y_{11} + \zeta_x & Y_2 &= y_{21} + \zeta_y \end{aligned} \quad (26)$$

where ζ_x and ζ_y are two lumped uncertainties that collect the slippage and external disturbances affecting the x and y channels, and ζ_x and ζ_y are an additive noise affecting the x and y measurement. The existence of ζ_x , ζ_y , ζ_x and ζ_y in a linearized obtained form represents an obstacle to ensure the convergence of the tracking error to zero. To deal with this problem, many researchers have proposed the Active Disturbance Rejection Control (ADRC) [18] as a solution. The principle of the ADRC approach is to estimate the effect of lumped uncertainties to the system by the extended state observer strategies and then compensate them. This concept is very complicated and difficult when considering important values of uncertainties. To overcome this problem, a new control algorithm based on the interval observer is proposed to guarantee the tracking results of the uncertain WMR.

V. INTERVAL OBSERVER DESIGN

In this section, interval observers are applied for the Burnovsky system defined by equation (26). In a matrix form, this latter is written as follows:

$$\begin{cases} \dot{x}_1 = A_1 x_1 + B_1 v_x + v_1 \\ Y_1 = C_1 x_1 + y_1 \end{cases}, \quad \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 v_y + v_2 \\ Y_2 = C_2 x_2 + y_2 \end{cases} \quad (27)$$

with $\bar{x}_1 = [x_{11}; x_{12}]^T$, $\underline{x}_2 = [x_{21}; x_{22}]^T$, $A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;

$$B_1 = B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T, \quad v_1 = 0 \quad v_x^T, \\ v_2 = 0 \quad v_y^T; C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The main idea of designing interval observers of the uncertain system (27) is to create lower and upper bounds of the real states $x_1(t)$ and $x_2(t)$, which allows us to guarantee that both latter belong to a specific interval. The observer interval design of system (27) requires the following assumption:

A1. Pairs $(A_1; C_1)$ and $(A_2; C_2)$ are observable.

A2. There exist L_1 and L_2 gains such that matrices $(A_1 - L_1 C_1)$ and $(A_2 - L_2 C_2)$ are Hurwitz and Metzler (off-diagonal elements are positive).

A3. The lumped uncertainties and the measurement noise are unknown but bounded with known bounds $v_{-1}, v_{-2}, v_{-x}, v_{-y}, v_{-1}, v_{-2}, v_{-x}$ and v_{-y} .

According to assumptions 1, 2 and 3, the interval observers of system (27) are defined as follows:

$$\begin{cases} \dot{\bar{x}}_{-1} = A_1 \bar{x}_{-1} + B_1 v_x + L_1 C_1 (\bar{x}_{-1} - \bar{y}_1) + \bar{v}_{-1} + L_1 v_{-x} \\ \dot{\underline{x}}_{-1} = A_1 \underline{x}_{-1} + B_1 v_x + L_1 C_1 (\underline{x}_{-1} - \underline{y}_1) + \underline{v}_{-1} + L_1 v_{-x} \\ \bar{x}_{-1}(0) \quad \underline{x}_{-1}(0) \quad \bar{y}_1(0) \quad \underline{y}_1(0) \end{cases} \quad (28)$$

$$\begin{cases} \dot{\bar{x}}_{-2} = A_2 \bar{x}_{-2} + B_2 v_y + L_2 C_2 (\bar{x}_{-2} - \bar{y}_2) + \bar{v}_{-2} + L_2 v_{-y} \\ \dot{\underline{x}}_{-2} = A_2 \underline{x}_{-2} + B_2 v_y + L_2 C_2 (\underline{x}_{-2} - \underline{y}_2) + \underline{v}_{-2} + L_2 v_{-y} \\ \bar{x}_{-2}(0) \quad \underline{x}_{-2}(0) \quad \bar{y}_2(0) \quad \underline{y}_2(0) \end{cases} \quad (29)$$

For system (27), it is impossible to compute gains L_1 and L_2 such that $(A_1 - L_1 C_1)$ and $(A_2 - L_2 C_2)$ are Metzler and Hurwitz, which presents an obstacle for the construction of the interval observers. This drawback is surmounted via a change in coordinates $Z_1 = G_1 x_1$ and $Z_2 = G_2 x_2$ such that $E_1 = G_1 H_{ob1} G_1^{-1}$ and $E_2 = G_2 H_{ob2} G_2^{-1}$ are Metzler and Hurwitz. The detail of calculating the transformation matrices G_1 and G_2 can be found in [14]. Actually, when introducing the new variables $Z_1 = G_1 x_1$ and $Z_2 = G_2 x_2$, system (27) can be presented as follows:

$$\begin{cases} \dot{Z}_1 = G_1 A_1 G_1^{-1} Z_1 + G_1 B_1 v_x + G_1 v_{-1} \\ Y_1 = C_1 G_1^{-1} Z_1 + y_1 \end{cases} \quad (30)$$

$$\begin{cases} \dot{Z}_2 = G_2 A_2 G_2^{-1} Z_2 + G_2 B_2 v_y + G_2 v_{-2} \\ Y_2 = C_2 G_2^{-1} Z_2 + y_2 \end{cases} \quad (31)$$

The interval observers of the new systems (30) and (31) are defined as follows:

$$\begin{cases} \dot{\bar{Z}}_1 = E_1 \bar{Z}_1 + G_1 B_1 v_x + G_1 v_{-1} & G_1 v_{-1} + G_1 L_1 Y_1 + G_1 L_1 v_{-x} \\ \dot{\underline{Z}}_1 = E_1 \underline{Z}_1 + G_1 B_1 v_x + G_1 v_{-1} & G_1 v_{-1} + G_1 L_1 Y_1 + G_1 L_1 v_{-x} \\ \bar{Z}_1(0) = G_1^{-1} \bar{x}_1(0) & G_1^{-1} \bar{x}_1(0) \\ \underline{Z}_1(0) = G_1^{-1} \underline{x}_1(0) & G_1^{-1} \underline{x}_1(0) \end{cases} \quad (32)$$

$$\begin{cases} \dot{\bar{Z}}_2 = E_2 \bar{Z}_2 + G_2 B_2 v_y + G_2 v_{-2} & G_2 v_{-2} + G_2 L_2 Y_2 + G_2 L_2 v_{-y} \\ \dot{\underline{Z}}_2 = E_2 \underline{Z}_2 + G_2 B_2 v_y + G_2 v_{-2} & G_2 v_{-2} + G_2 L_2 Y_2 + G_2 L_2 v_{-y} \\ \bar{Z}_2(0) = G_2^{-1} \bar{x}_2(0) & G_2^{-1} \bar{x}_2(0) \\ \underline{Z}_2(0) = G_2^{-1} \underline{x}_2(0) & G_2^{-1} \underline{x}_2(0) \end{cases} \quad (33)$$

with $E_1 = G_1(A_1 - L_1 C_1)G_1^{-1}$; $G_1^+ = \max(0; G_1)$; $G_1^- = G_1^+ - G_1$, $E_2 = G_2(A_2 - L_2 C_2)G_2^{-1}$; $G_2^+ = \max(0; G_2)$; $G_2^- = G_2^+ - G_2$;

After designing the interval observers of the new systems (30) and (31), we can deduce the upper and lower states of the original systems (27) in the following way:

$$\begin{cases} \bar{x}_{-1} = R_1^+ \bar{Z}_1 & R_1^- \underline{Z}_1 & \bar{x}_{-2} = R_2^+ \bar{Z}_2 & R_2^- \underline{Z}_2 \\ \underline{x}_{-1} = R_1^- \bar{Z}_1 & R_1^+ \underline{Z}_1 & \underline{x}_{-2} = R_2^- \bar{Z}_2 & R_2^+ \underline{Z}_2 \\ \bar{x}_{-1}(0) & \underline{x}_{-1}(0) & \bar{x}_{-2}(0) & \underline{x}_{-2}(0) \end{cases} \quad (34)$$

with $R_1 = G_1^{-1}$; $R_1^+ = \max(0; R_1)$; $R_1^- = R_1^+ - R_1$, $R_2 = G_2^{-1}$; $R_2^+ = \max(0; R_2)$; $R_2^- = R_2^+ - R_2$.

Consider the error equations $\bar{e}_{z1} = \bar{Z}_1 - Z_1$, $\underline{e}_{z1} = Z_1 - \underline{Z}_1$, $\bar{e}_{z2} = \bar{Z}_2 - Z_2$ and $\underline{e}_{z2} = Z_2 - \underline{Z}_2$. The dynamics of upper and lower observer errors are defined as follows:

$$\begin{cases} E_{Obx} \quad \dot{\bar{e}}_{z1} = E_1 \bar{e}_{z1} + v_{-z1} & E_{Oby} \quad \dot{\bar{e}}_{z2} = E_2 \bar{e}_{z2} + v_{-z2} \\ \dot{\underline{e}}_{z1} = E_1 \underline{e}_{z1} + v_{-z1} & \dot{\underline{e}}_{z2} = E_2 \underline{e}_{z2} + v_{-z2} \end{cases} \quad (35)$$

with $\bar{z}_{-1} = G_1^{-1} \bar{x}_{-1} - G_1^{-1} \bar{y}_1$, $\underline{z}_{-1} = G_1^{-1} \underline{x}_{-1} - G_1^{-1} \underline{y}_1$; $\bar{z}_{-2} = G_2^{-1} \bar{x}_{-2} - G_2^{-1} \bar{y}_2$, $\underline{z}_{-2} = G_2^{-1} \underline{x}_{-2} - G_2^{-1} \underline{y}_2$ and $\bar{z}_{-2} = G_2^{-1} \bar{x}_{-2} - G_2^{-1} \bar{y}_2$, $\underline{z}_{-2} = G_2^{-1} \underline{x}_{-2} - G_2^{-1} \underline{y}_2$.

Since E_1 and E_2 are assumed to be non-negative, \bar{z}_{-1} , \underline{z}_{-1} , \bar{z}_{-2} and \underline{z}_{-2} are non-negative. As a result, for any initial condition $\bar{Z}_1(0)$, $\underline{Z}_1(0)$, $\bar{Z}_2(0)$ and $\underline{Z}_2(0)$ chosen such that $\bar{e}_{z1}(0)$, $\underline{e}_{z1}(0)$, $\bar{e}_{z2}(0)$ and $\underline{e}_{z2}(0)$ are non-negative, the dynamics of the interval estimation errors \bar{e}_{z1} , \underline{e}_{z1} , \bar{e}_{z2} and \underline{e}_{z2} stay always non-negative for all time t , so the state is bounded as follows:

$$\underline{Z}_1(t) \leq Z_1(t) \leq \bar{Z}_1(t); \underline{Z}_2(t) \leq Z_2(t) \leq \bar{Z}_2(t) \quad (36)$$

Moreover, functions \bar{z}_{-1} , \underline{z}_{-1} , \bar{z}_{-2} and \underline{z}_{-2} are globally Lipschitz, and consequently for $\bar{Z}_1 - Z_1 - \bar{Z}_1$ and $\underline{Z}_2 - Z_2 - \underline{Z}_2$ and for a chosen submultiplicative norm, there exist positive constants α_{z1i} and α_{z2i} $i = 1:6$ such that:

$$|\bar{z}_{-1} - z_{1j}| \leq \alpha_{z11} |\bar{Z}_1 - Z_1| + \alpha_{z12} |\bar{Z}_1 - Z_1| + \alpha_{z13} \quad (37)$$

$$|\underline{z}_{-1} - z_{1j}| \leq \alpha_{z14} |\underline{Z}_1 - Z_1| + \alpha_{z15} |\underline{Z}_1 - Z_1| + \alpha_{z16} \quad (38)$$

$$|\bar{z}_{-2} - z_{2j}| \leq \alpha_{z21} |\bar{Z}_2 - Z_2| + \alpha_{z22} |\bar{Z}_2 - Z_2| + \alpha_{z23} \quad (39)$$

$$|\underline{z}_{-2} - z_{2j}| \leq \alpha_{z24} |\underline{Z}_2 - Z_2| + \alpha_{z25} |\underline{Z}_2 - Z_2| + \alpha_{z26} \quad (40)$$

According to [19], if there exist positive definite and symmetric matrices P_{x2} , P_{y2} , Q_{x2} , Q_{y2} , α and β such that the following Linear Matrix Inequality (LMI) are satisfied:

$$\begin{aligned} E_{z1}^T P_{x2} + P_{x2} E_{z1} + \alpha P_{x2}^2 + \frac{\alpha^2}{x} I_1 + Q_{x2} & \begin{matrix} P_{x2} \\ \frac{I_1}{x} \end{matrix} & 0; \\ \text{"} & \# & (41) \\ E_{z2}^T P_{y2} + P_{y2} E_{z2} + \beta P_{y2}^2 + \frac{\beta^2}{y} I_2 + Q_{y2} & \begin{matrix} P_{y2} \\ \frac{I_2}{y} \end{matrix} & 0; \end{aligned} \quad (42)$$

with $E_{z1} = \text{diag}(E_1; E_1)$, $E_{z2} = \text{diag}(E_2; E_2)$, $\alpha = 2 \max(\frac{2}{z_{11}}; \frac{2}{z_{12}}; \frac{2}{z_{13}}; \frac{2}{z_{14}}; \frac{2}{z_{15}}; \frac{2}{z_{16}})$ and $\beta = 2 \max(\frac{2}{z_{21}}; \frac{2}{z_{22}}; \frac{2}{z_{23}}; \frac{2}{z_{24}}; \frac{2}{z_{25}}; \frac{2}{z_{26}})$. Then, variables $\bar{z}_1(t)$, $\underline{z}_1(t)$, $\bar{z}_2(t)$ and $\underline{z}_2(t)$ are bounded for all the time. The interval observer design enables obtaining deterministic dynamic intervals containing a real state vector. Accordingly, the centre of the interval can be chosen as a robust state estimation for the uncertain WMR as follows:

$$\hat{z}_1 = \frac{-1 + \bar{z}_1}{2}; \hat{z}_2 = \frac{-2 + \bar{z}_2}{2}; \hat{z} = \frac{+}{2}; \hat{v} = \frac{V + \bar{v}}{2} \quad (43)$$

Based on equations (5) and (6), the lower and upper values of \hat{z} and \hat{v} can be deduced as follows:

$$\underline{\hat{z}} = \arctan(\frac{-22}{12}); \bar{\hat{z}} = \arctan(\frac{-22}{-12}) \quad (44)$$

$$\underline{\hat{v}} = \frac{q}{-12 + \frac{2}{22}}; \bar{\hat{v}} = \frac{q}{-12 + \frac{-2}{22}} \quad (45)$$

Choosing the center of the interval as a robust state estimation has no influence on the trajectory tracking results because any $\hat{z} \in [\underline{\hat{z}}; \bar{\hat{z}}]$ can be considered as a guaranteed state estimation for the WMR. Generally, the center of the interval is a classic choice used in many researchers [11,20] studying the control based on the interval observer.

VI. GUARANTEED TRACKING CONTROL

In this section, based on the interval observer result, a guaranteed tracking control is developed for the uncertain WMR. Replacing the real state by a robust state estimation defined by equation (43) in the feedback controllers (12) and (13), the estimated feedback law can be obtained as follows:

$$\hat{v}_x = \alpha_{xd} \quad K_{x2} \hat{e}_{rx1} \quad K_{x1} \hat{e}_{rx1} \quad (46)$$

$$\hat{v}_y = \alpha_{yd} \quad K_{y2} \hat{e}_{ry1} \quad K_{y1} \hat{e}_{ry1} \quad (47)$$

with $\hat{e}_{rx1} = \hat{z}_{11} - x_d$ and $\hat{e}_{ry1} = \hat{z}_{21} - y_d$. When replacing state \hat{z} by the robust state estimation \hat{z} in the U_{FBTC} control (17), the Guaranteed Flatness-Based Tracking Control (GFBTC) applied to the uncertain WMR system can be obtained as follows:

$$\begin{aligned} U_{GFBTC1} &= \hat{B}^{-1} \alpha_{xd} \quad K_{x2} \hat{e}_{rx1} \quad K_{x1} \hat{e}_{rx1} \\ U_{GFBTC2} & \quad \alpha_{yd} \quad K_{y2} \hat{e}_{ry1} \quad K_{y1} \hat{e}_{ry1} \end{aligned} \quad (48)$$

U_{GFBTC} differs from U_{FBTC} in the way that in U_{GFBTC} the robust state estimation \hat{z} is utilized, but in U_{FBTC} state z is applied. To analyze the stability of the WMR system, the stability of the tracking error dynamics of positions x and y must be studied. To prove the stability of the tracking error

dynamics of position x , let us define the Lyapunov function candidate as follows:

$$V_{r1} = e_{rx}^T P_{x1} e_{rx} \quad (49)$$

where $e_{rx} = [e_{rx1}; e_{rx1}]^T$, and P_{x1} is a symmetric positive definite matrix. The derivative of the Lyapunov function V_{r1} is defined as follows:

$$\dot{V}_{r1} = e_{rx}^T P_{x1} \dot{e}_{rx} + e_{rx}^T P_{x1} \dot{e}_{rx} \quad (50)$$

When substituting \dot{e}_{rx} by equation (18) in equation (50), the derivative of the Lyapunov function V_{r1} is defined as follows:

$$\dot{V}_{r1} = e_{rx}^T (H_{rx}^T P_{x1} + P_{x1} H_{rx}) e_{rx} \quad (51)$$

Gains k_{x1} and k_{x2} are chosen such that matrix H_{rx} is Hurwitz. Thereby, for any symmetric positive definite matrix Q_{x1} , there exists a symmetric positive definite matrix P_{x1} satisfying the Lyapunov equation:

$$H_{rx}^T P_{x1} + P_{x1} H_{rx} = -Q_{x1} \quad (52)$$

According to (52), the derivative of the Lyapunov function V_{r1} can be written as follows:

$$\dot{V}_{r1} = -e_{rx}^T Q_{x1} e_{rx} \quad (53)$$

Thus, based on the Lyapunov method, the asymptotic stability for trajectory tracking is deduced.

To prove the stability of the error observer of the x position, let us recall the proof defined in [19], which shows that variables $\bar{z}_1(t)$ and $\underline{z}_1(t)$ are bounded. Let Consider the positive definite quadratic Lyapunov function as follows:

$$V_{z1} = e_{z1}^T P_{x2} e_{z1} \quad (54)$$

where $e_{z1} = [e_{z1}^T; e_{z1}^T]^T$, and P_{x2} is a positive definite symmetric matrix. The observation error of system E_{obx} can be rewritten as:

$$\dot{e}_{z1} = E_{z1} e_{z1} + \alpha_{z1} \quad (55)$$

with $\alpha_{z1} = [\alpha_{z1}^T; \alpha_{z1}^T]^T$. Due that matrix E_{z1} is Hurwitz and Metzler, so is the matrix E_{z1} . The derivative of V_{z1} can be defined as follows:

$$\dot{V}_{z1} = e_{z1}^T (E_{z1}^T P_{x2} + P_{x2} E_{z1}) e_{z1} + 2e_{z1}^T P_{x2} \alpha_{z1} \quad (56)$$

$$e_{z1}^T (E_{z1}^T P_{x2} + P_{x2} E_{z1}) e_{z1} + \alpha_{z1}^T P_{x2}^2 e_{z1} + \frac{1}{x} \alpha_{z1}^T \alpha_{z1} \quad (57)$$

According to Corollary 1 defined in [19], there exists a positive constant α_{z1} such that:

$$\alpha_{z1}^T \alpha_{z1} \leq \alpha_{z1} (j e_{z1}^T j^2 + 1) \quad (58)$$

with $\alpha_{z1} = 2 \max(\frac{2}{z_{11}}; \frac{2}{z_{21}}; \frac{2}{z_{31}}; \frac{2}{z_{41}}; \frac{2}{z_{51}}; \frac{2}{z_{61}})$. Thus, equation (57) can be written as:

$$\dot{V}_{z1} = e_{z1}^T (E_{z1}^T P_{x2} + P_{x2} E_{z1} + \alpha_{z1} P_{x2}^2 + \frac{\alpha_{z1}^2}{x} I_1) e_{z1} + \alpha_{z1} \quad (59)$$

Therefore, if there exist positive definite symmetric matrices P_{x2} and Q_{x2} and a positive scalar α_{z1} such that the LMI defined in (41) is satisfied, then equation (59) can be defined as follows:

$$\dot{V}_{z1} = -e_{z1}^T (Q_{x2}) e_{z1} + \alpha_{z1} \quad (60)$$

This implies that variables $\underline{z}_1(t)$ and $\bar{z}_1(t)$ are bounded for all $t_0 \geq 0$. To prove the global stability of the complete closed-loop system (system + controller + state observer), let us define the following Lyapunov function candidate:

$$V_1 = V_{r1} + V_{z1} \quad (61)$$

It follows from the asymptotic stability of each subsystem that the global asymptotic stability of the error dynamics of position x is guaranteed. The stability of the error dynamics of position y can be studied in the same way as position x .

VII. SIMULATION AND RESULTS

In this section, two kinds of simulation are presented to validate the performance obtained by the guaranteed tracking control. The parameters of the considered WMR are given by: $r = 0.1 \text{ m}$, $b = 0.15 \text{ m}$. A reference trajectory is generated for the WMR permitting its movement from the initial state $X(0) = [0; 0; \bar{2}; 45]^T$ to the final state $X(5) = [4; 7; \bar{2}; 45]^T$. In addition, the obtained trajectory must respect the following constraint:

$$\begin{aligned} & 0 \text{ m} \leq x_d \leq 5 \text{ m}; 0 \text{ m} \leq y_d \leq 10 \text{ m}; \\ & 90^\circ \leq \alpha_d \leq 90^\circ; 0.1 \text{ m/s} \leq v_d \leq 10 \text{ m/s}; \\ & v_x(0) = v_y(0) = v_x(5) = v_y(5) = 1 \text{ m/s} \end{aligned} \quad (62)$$

An 11-order Bezier curve for each flat output is considered to generate the path for the WMR, which can be formulated as follows:

$$x_d(t) = \sum_{k=1}^{11} B_{k,11}(t) x_k; \quad y_d(t) = \sum_{k=1}^{11} B_{k,11}(t) y_k \quad (63)$$

where x_k and y_k , $k = 0; \dots; 11$ are the control parameters of the Bezier curve, and $B_{k,11}(t)$ is a Bernstein polynomial defined as follows:

$$B_{k,11}(t) = \frac{11!}{k!(11-k)!} \left(\frac{t_f}{t_f-t_0}\right)^k \left(\frac{t-t_0}{t_f-t_0}\right)^{11-k} \quad (64)$$

A. Simulation 1

In this simulation, the lower and upper values of the initial state and all the uncertainties are taken as $\underline{x}_1(0) = [2; 0.2]^T$; $\underline{x}_2(0) = [2; 0.2]^T$; $\bar{x}_1(0) = [4; 4.2]^T$; $\bar{x}_2(0) = [4; 7.47]^T$; $\underline{d}_1 = \underline{d}_2 = \underline{d}_3 = \underline{d}_4 = \underline{d}_5 = \underline{d}_6 = 0.3$; $\bar{d}_1 = \bar{d}_2 = \bar{d}_3 = \bar{d}_4 = \bar{d}_5 = \bar{d}_6 = 0.25$; $\underline{d}_7 = \underline{d}_8 = \underline{d}_9 = \underline{d}_{10} = \underline{d}_{11} = 0.1$; $\bar{d}_7 = \bar{d}_8 = \bar{d}_9 = \bar{d}_{10} = \bar{d}_{11} = 0.2$; $\underline{x}_x = \underline{x}_y = 0.5$; $\bar{x}_x = \bar{x}_y = 1.5$. The controller design parameters are chosen as $\omega_c = \omega_y = 5 \text{ rad/s}$ and $\omega_x = \omega_y = 1$. When choosing the matrix gain $L_1 = L_2 = 25 \text{ } 150^T$; matrices $(A_1 \text{ } L_1 C_1)$ and $(A_2 \text{ } L_2 C_2)$ are Hurwitz and have the eigenvalues -10 and -15 . This choice of eigenvalues makes the observer dynamics faster than the system. According to [14], matrices G_1 and G_2 , chosen such that matrices E_1 and E_2 are Metzler and Hurwitz, are defined as $G_1 = G_2 = \begin{bmatrix} 2 & 0.2 \\ 3 & 0.2 \end{bmatrix}$.

Take $z_{1i} = z_{2i} = 1$, $i = 1; \dots; 6$, so $z_1 = z_2 = 2$. By solving the LMI defined in (41) and (42), we find that $\omega_x = \omega_y = 1$, $P_{x2} = \text{diag}(P_{x21}; P_{x21})$, $P_{y2} = \text{diag}(P_{y21}; P_{y21})$, $Q_{x2} = \text{diag}(Q_{x21}; Q_{x21})$ and $Q_{y2} = \text{diag}(Q_{y21}; Q_{y21})$, where Q_{x21} and Q_{y21} are the identity matrices 2×2 , and P_{x21}

$$\text{and } P_{y21} \text{ are defined as } P_{x21} = P_{y21} = \begin{bmatrix} 0.1650 & 0 \\ 0 & 0.1076 \end{bmatrix}.$$

In order to show the robustness of the proposed control, FBTC and GFBTC are applied to the uncertain WMR system under the same conditions. As a consequence, the WMR first starts from an uncertain initial condition $\hat{X}(0) \in [\underline{X}(0); \bar{X}(0)]$ defined as $\hat{X}(0) = [x(0); y(0); \vartheta(0); \dot{x}(0)]^T = [1; 1; 7.74; 60]^T$. Second, it is subjected to unknown uncertainties with known bounds defined previously. Figure 2 depicts the following results:

- The red curves represent the desired trajectory of the WMR.
- The blue and black curves illustrate the upper and lower state estimations of the uncertain WMR system.
- The pink and green curves show the tracking results when applying FBTC and GFBTC, respectively, to the uncertain WMR system.

From Figure 2, it can be firstly observed that choosing a properly gain enables the interval observer to provide a time-varying enclosure, which contains all possible real state vectors of the uncertain WMR. This type of observers allows guaranteeing the estimation result of a WMR despite the presence of bounded uncertainties. In addition, it can be demonstrated that exploiting the estimation result obtained by the interval observer in the design of GFBTC permits the WMR to move in a precise interval containing the desired reference trajectory. Whereas, when applying FBTC to the uncertain WMR system, this latter diverges strongly from the reference. As a result, the controllers that are not based on an uncertain model, even if they are feedback controllers, may not work correctly.

B. Simulation 2

A second simulation collection is carried out with the same references and the same bound values of the initial state considered in simulation 1 and with other important values of uncertainty conditions for checking robustness. In this simulation, we assume that the WMR is subjected to unknown uncertainties with known bounds defined as $\underline{d}_1 = \underline{d}_2 = \underline{d}_3 = \underline{d}_4 = \underline{d}_5 = \underline{d}_6 = 0.7$; $\bar{d}_1 = \bar{d}_2 = \bar{d}_3 = \bar{d}_4 = \bar{d}_5 = \bar{d}_6 = 0.6$; $\underline{d}_7 = \underline{d}_8 = \underline{d}_9 = \underline{d}_{10} = \underline{d}_{11} = 0.5$; $\bar{d}_7 = \bar{d}_8 = \bar{d}_9 = \bar{d}_{10} = \bar{d}_{11} = 0.5$; $\underline{x}_x = \underline{x}_y = 0.5$; $\bar{x}_x = \bar{x}_y = 1.5$. The feedback gain of the controllers and the observer have been adjusted to obtain a smooth and fast tracking performance. The tracking performances related to simulation 2 are illustrated in Figure 3 where comparative simulation with FBTC and GFBTC is also given. From Figure 3, the effectiveness of GFBTC can be seen compared to FBTC. Furthermore, the width of the estimated interval increases compared the interval width obtained in simulation 1, and this is due to the augmentation of the value of uncertainties. In spite of this disadvantage, it can be observed that the WMR still moves in a precise interval containing the reference trajectory. Consequently, it can be deduced that the combination between the flatness control and the interval observer permits obtaining a guaranteed guidance law for the WMR despite the existence of unknown uncertainties with important bound values.

